Spin chains with electrons in Penning traps

G. Ciaramicoli, I. Marzoli, and P. Tombesi

Dipartimento di Fisica, Università degli Studi di Camerino, 62032 Camerino, Italy

(Dated: February 1, 2008)

Abstract

We demonstrate that spin chains are experimentally feasible using electrons confined in micro-Penning traps, supplemented with local magnetic field gradients. The resulting Heisenberg-like system is characterized by coupling strengths showing a dipolar decay. These spin chains can be used as a channel for short distance quantum communication. Our scheme offers high accuracy in reproducing an effective spin chain with relatively large transmission rate.

PACS numbers: 03.65.-w, 03.67.-a, 03.67.Hk, 03.67.Lx

I. INTRODUCTION

Recently much theoretical research work has been focused on the possibility to use systems of spins, coupled by ferromagnetic Heisenberg interactions and arranged along chain structures, for transferring quantum information. The remarkable property of these systems is the capability of transmitting the qubit state along the chain with fidelity exceeding the classical threshold and by means only of their free dynamical evolution. After the seminal paper by Bose [1], in which the potentialities of the so-called spin chains have been shown, several strategies have been proposed to increase the transmission fidelity [2] and even to achieve, under appropriate conditions, perfect state transfer [3, 4, 5, 6]. All these proposals refer to ideal spin chains in which only nearest-neighbor couplings are present. However, also the more realistic case of long range couplings, in particular magnetic dipole like couplings, has been studied [7, 8]. In [7] it has been shown that perfect state transfer or, at least, high transmission fidelity can be obtained by appropriately choosing the system parameters, such as local magnetic fields and inter-spin distances. Moreover, even when no site specific locally-tunable fields are allowed, spin chains with dipolar couplings often perform better, in terms of transmission fidelity, than their nearest-neighbor coupled counterpart [8]. Hence, from these theoretical predictions, we expect that spin chains, also in the case of long range interactions, may represent a very promising system to achieve high fidelity quantum information transfer without requiring experimentally demanding gating operations.

In this paper we demonstrate that a linear array of electrons, confined in micro-Penning traps, can implement an effective spin chain with magnetic dipole like spin coupling. The same system consisting of trapped electrons in vacuum has been already proposed as a valid and competitive candidate for universal quantum information processing [9, 10, 11]. In this respect, the possibility of reliably transmitting the qubit state between different quantum registers, without applying gate operations, is highly desirable. In fact, the use of a quantum channel to transfer a qubit state in a quantum processor can be a valuable alternative to the repeated application of swapping gates.

We have already proved in [11] that the addition of a magnetic field gradient, together with the Coulomb interaction between the particles, allows to obtain an effective nuclear magnetic resonance (NMR)-like coupling between the spins of the confined electrons. Here we generalize this approach to encompass a variety of trap set up, also in connection with

novel geometries of Penning traps [12, 13]. Indeed, by further investigating the interaction between the internal (spin) and external (motional) degrees of freedom of each particle, introduced by the applied local magnetic field gradient, we can mimic more general systems, with Heisenberg ferromagnetic or antiferromagnetic Hamiltonian. This fact opens up the possibility to simulate quantum spin systems with tunable interactions, thanks to the experimental control over the different trap parameters. The ultimate goal may be the observation of quantum phase transitions, as proposed with trapped ions controlled by laser beams [14].

In our proposal, we consider a linear array of electrons with inter-particle distance ranging from few microns to 50 μm. We provide an analytical expression for the spin-spin coupling strength, which shows a dipolar decay law. We estimate the value of the spin-spin coupling, for different ranges of the system characteristic frequencies as well as of the intensity of the magnetic gradient, with the aim of optimizing the transfer time of our quantum channel. Furthermore, we evaluate the fidelity of our system in reproducing an effective spin chain according to the Heisenberg model. In particular, we calculate the probability to obtain a perfect spin state transfer in a chain consisting of just two electrons. This probability, equal to one in an ideal spin chain [1], in our system is less than one owing to the effects resulting from the interplay between the internal and the external degrees of freedom of the trapped particles. However, by an appropriate choice of the system parameters, especially the frequency and the amplitude of the spatial motions, we can obtain high fidelities and, at the same time, sufficiently large values of the spin-spin coupling. The electron trapping arrangement offers also the possibility to apply arbitrary site-specific changes in the system parameters in order to maximize, as outlined in [7, 8], the efficiency of the quantum channel. Our theoretical predictions suggest that a linear array of electrons is suitable to implement a spin chain with the present technology.

The manuscript is organized as follows. In Sec. II we describe the system and how the local magnetic field gradient couples the electron spin to the motional degrees of freedom. This coupling, mediated by the Coulomb interaction between charged particles, results in an effective Heisenberg-like Hamiltonian (Sec. III). In Sec. IV we estimate the fidelity and the efficiency of our system as a channel for quantum information transmission. Finally, the results of our analysis are summarized and discussed in Sec. V. The more technical details, concerning the derivation of the fidelity, are presented in Appendix A.

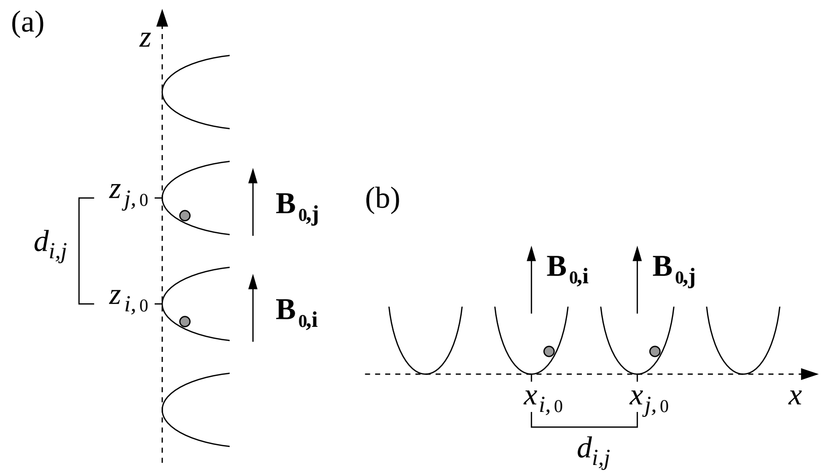

FIG. 1: Schematic drawing illustrating two different geometries for a linear array of micro-Penning traps. The traps are represented by sketching the electrostatic potential along the z axis. (a) The electrons are aligned along the z axis, parallel to the confining magnetic field; (b) the electrons are aligned along the x axis, orthogonal to the confining magnetic field.

II. A LINEAR ARRAY OF TRAPPED ELECTRONS

We consider a linear array of N electrons in micro-Penning traps [15]. According to the different geometry of the electrode arrangement, the micro-trap array can be either parallel to the direction of the confining magnetic field, i.e. along the z axis as shown in Fig. 1(a), or orthogonal to this field, for example, along the x axis as shown in Fig. 1(b). To confine electrons in an array along the z direction we can use a closed cylindrical electrode structure [9, 10] or an open wire arrangement [12]. This latter structure can also accommodate the electrons in an array aligned along the x axis. An orderly set of micro-traps, orthogonal to the trapping magnetic field, can be likewise realized by means of a planar electrode system [13]. As we will see, the different orientation of the linear array of particle affects the form of the resulting interaction Hamiltonian. Hence, we firstly derive the expression of the effective Hamiltonian in the case of micro-traps aligned along the z axis. Then we will show how this expression modifies in the case of an array directed along the x axis.

The Hamiltonian of a system of N electrons confined in an array of Penning traps can be written as

$$H = \sum_{i=1}^{N} H_i^{NC} + \sum_{i<j} H_{i,j}^{C}, \tag{1}$$

where

$$H_i^{NC} = \frac{(\mathbf{p_i} - e\mathbf{A_i})^2}{2m_e} + eV_i - \frac{ge\hbar}{4m_e}\boldsymbol{\sigma_i}\cdot\mathbf{B_i} \tag{2}$$

represents the single electron dynamics inside each trap and

$$H_{i,j}^C = \frac{e^2}{4\pi\epsilon_0|\mathbf{r_i} - \mathbf{r_j}|} \tag{3}$$

describes the Coulomb interaction between electrons i and j. In Eqs. (2) and (3) m_e, e, g, and $\boldsymbol{\sigma_i}$ are, respectively, the electron mass, charge, giromagnetic factor, and Pauli spin operators. As shown in Fig. 1(a), we assume that the micro-traps are aligned along the z axis and that $z_{i,0}$ is the position of the center of the i-th trap. The electrostatic potential

$$V_i(x_i, y_i, z_i) \equiv V_0\,\frac{(z_i - z_{i,0})^2 - (x_i^2 + y_i^2)/2}{\ell^2} \tag{4}$$

is the usual quadrupole potential of a Penning trap, where V_0 is the applied potential difference between the trap electrodes and ℓ is a characteristic trap length. The magnetic field

$$\mathbf{B_i} \equiv -\frac{b}{2}(x_i\,\mathbf{i} + y_i\,\mathbf{j}) + [B_0 + b(z_i - z_{i,0})]\,\mathbf{k} \tag{5}$$

is the sum of the uniform magnetic field $B_0\,\mathbf{k}$, providing the radial confinement, with a local linear magnetic gradient around the i-th trap. The associated vector potential

$$\mathbf{A_i} \equiv \frac{1}{2}[B_0 + b(z_i - z_{i,0})](-y_i\,\mathbf{i} + x_i\,\mathbf{j}) \tag{6}$$

preserves the cylindrical symmetry of the unperturbed trapping field.

Following an approach similar to the one described in Ref. [11], the spatial part of H_i^{NC} can be recast in the form

$$\begin{aligned} H_i^{(ext)} \;\simeq\; & -\hbar\omega_m a_{m,i}^\dagger a_{m,i} + \hbar\omega_c a_{c,i}^\dagger a_{c,i} + \hbar\omega_z a_{z,i}^\dagger a_{z,i} \\ & +\; \hbar\omega_z\varepsilon(a_{z,i} + a_{z,i}^\dagger)\left(\frac{\omega_m}{\omega_c}a_{m,i}^\dagger a_{m,i} + a_{c,i}^\dagger a_{c,i}\right), \end{aligned} \tag{7}$$

where the annihilation operators $a_{m,i}$, $a_{c,i}$, $a_{z,i}$ [11, 16] refer, respectively, to the magnetron, cyclotron and axial oscillators of the i-th electron. The frequencies of these oscillators

$$\omega_m \simeq \frac{\omega_z^2}{2\omega_c}, \tag{8}$$

$$\omega_c \simeq \frac{|e|B_0}{m_e}, \tag{9}$$

$$\omega_z = \sqrt{\frac{2eV_0}{m_e\ell^2}} \tag{10}$$

depend on the applied external fields and on the trap size. They build up a well defined hierarchy with $\omega_m \ll \omega_z \ll \omega_c$. Indeed, we exploit this fact together with the assumption of a weak magnetic gradient, such that $b|z_i - z_{i,0}|/B_0 \ll 1$, to derive the Hamiltonian (7). The dimensionless parameter

$$\varepsilon \equiv \frac{|e|b}{m_e\omega_z}\sqrt{\frac{\hbar}{2m_e\omega_z}} = \frac{|e|b\Delta z}{m_e\omega_z}, \tag{11}$$

with Δz being the ground state amplitude of the axial oscillator, represents the coupling, due to the magnetic gradient, between the axial motion and the radial degrees of freedom. In a similar way the magnetic gradient introduces also an interaction between the spatial and the spin motion. This coupling becomes evident by considering the spin part of the Hamiltonian Eq. (2)

$$\begin{aligned} H_i^{(spin)} &\equiv -\frac{ge\hbar}{4m_e}\boldsymbol{\sigma_i} \cdot \mathbf{B_i} \\ &= \frac{g\hbar}{4}\omega_c\sigma_i^z + \frac{g\hbar|e|b}{4m_e}\sigma_i^z(z_i - z_{i,0}) - \frac{g\hbar|e|b}{8m_e}(\sigma_i^x x_i + \sigma_i^y y_i) \end{aligned} \tag{12}$$

which, in terms of the ladder operators, becomes [11]

$$H_i^{(spin)} \simeq \frac{\hbar}{2}\omega_s\sigma_i^z + \frac{g}{4}\varepsilon\hbar\omega_z\sigma_i^z(a_{z,i} + a_{z,i}^\dagger) - \frac{g}{4}\varepsilon\hbar\omega_z\sqrt{\frac{\omega_z}{\tilde{\omega}_c}}\left[\sigma_i^{(+)}(a_{c,i} + a_{m,i}^\dagger) + \sigma_i^{(-)}(a_{c,i}^\dagger + a_{m,i})\right], \tag{13}$$

where $\tilde{\omega}_c \equiv \sqrt{\omega_c^2 - 2\omega_z^2}$ is essentially a modified cyclotron frequency due to the insertion of the quadrupole potential. In deriving Eq. (13) we defined the operators $\sigma_i^{(\pm)} \equiv (\sigma_i^x \pm i\sigma_i^y)/2$ and the spin precession frequency $\omega_s \equiv (g/2)\omega_c$.

Hence, the Hamiltonian, Eq. (2), of a single electron can be written as

$$\begin{aligned} H_i^{NC} &\simeq -\hbar\omega_m a_{m,i}^\dagger a_{m,i} + \hbar\omega_c a_{c,i}^\dagger a_{c,i} + \hbar\omega_z a_{z,i}^\dagger a_{z,i} + \frac{\hbar}{2}\omega_s\sigma_i^z \\ &+ \hbar\omega_z\varepsilon(a_{z,i} + a_{z,i}^\dagger)\left(a_{c,i}^\dagger a_{c,i} + \frac{\omega_m}{\omega_c}a_{m,i}^\dagger a_{m,i} + \frac{g}{4}\sigma_i^z\right) \\ &- \frac{g}{4}\varepsilon\hbar\omega_z\sqrt{\frac{\omega_z}{\tilde{\omega}_c}}\left[\sigma_i^{(+)}(a_{c,i} + a_{m,i}^\dagger) + \sigma_i^{(-)}(a_{c,i}^\dagger + a_{m,i})\right]. \end{aligned} \tag{14}$$

We now assume that, for each particle of the array, the cyclotron oscillator is in the ground state and the magnetron oscillator is in a thermal state with an average excitation number $\bar{l} \ll \omega_c/\omega_m$. We recall that the ground state cooling of the cyclotron motion for electrons [17] and the reduction of the magnetron motion excitation number for electrons [16] and ions [18] have been experimentally obtained. Under these conditions, we can neglect in Eq. (14) the coupling between the axial oscillator and the radial motion. We can further simplify

Eq. (14) by means of the rotating wave approximation (RWA). Indeed, terms like $\sigma_i^{(+)} a_{m,i}^\dagger$ are rotating at a frequency $\omega_s - \omega_m$ much larger than the anomaly frequency $\omega_a \equiv \omega_s - \omega_c$ typical of terms like $\sigma_i^{(+)} a_{c,i}$ and, therefore, are negligible in RWA. Hence, the Hamiltonian Eq. (14) becomes

$$\begin{aligned} H_i^{NC} &\simeq -\hbar\omega_m a_{m,i}^\dagger a_{m,i} + \hbar\omega_c a_{c,i}^\dagger a_{c,i} + \hbar\omega_z a_{z,i}^\dagger a_{z,i} + \frac{\hbar}{2}\omega_s \sigma_i^z \\ &+ \frac{g}{4}\varepsilon\hbar\omega_z \left(a_{z,i} + a_{z,i}^\dagger\right)\sigma_i^z - \frac{g}{4}\varepsilon\hbar\omega_z \sqrt{\frac{\omega_z}{\tilde{\omega}_c}} \left(\sigma_i^{(+)} a_{c,i} + \sigma_i^{(-)} a_{c,i}^\dagger\right). \end{aligned} \tag{15}$$

We see that the applied magnetic field gradient couples the different electron spin components to the axial and to the cyclotron oscillators.

Let us consider the part of the Hamiltonian Eq. (3) describing the Coulomb interaction between two electrons trapped at the sites i and j. If the oscillation amplitude of the two electrons is much smaller than the inter-trap distance $d_{i,j} \equiv |z_{i,0} - z_{j,0}|$, we can expand the interaction Hamiltonian in a power series and retain the terms up to the second order

$$H_{i,j}^C \simeq -\frac{e^2}{4\pi\epsilon_0 d_{i,j}^2}(\tilde{z}_i - \tilde{z}_j) + \frac{e^2}{8\pi\epsilon_0 d_{i,j}^3}\left[2(\tilde{z}_i - \tilde{z}_j)^2 - (x_i - x_j)^2 - (y_i - y_j)^2\right], \tag{16}$$

where $\tilde{z}_i \equiv z_i - z_{i,0}$. The Coulomb interaction produces three effects on the electron dynamics: (i) a displacement of the equilibrium position along the z axis, (ii) a shift of the axial resonance frequency, and (iii) a coupling between the motional degrees of freedom of different particles. The first two effects are rather small and can be taken into account by redefining the trap center position and the corresponding axial frequency. Therefore, in the remainder of this section we focus on the coupled dynamics of the two electrons

$$\begin{aligned} H_{i,j}^C &\simeq -\frac{e^2}{4\pi\epsilon_0 d_{i,j}^3}(2\tilde{z}_i\tilde{z}_j - x_i x_j - y_i y_j) \\ &= -2\hbar\xi_{i,j}(a_{z,i} + a_{z,i}^\dagger)(a_{z,j} + a_{z,j}^\dagger) \\ &+ \hbar\xi_{i,j}\frac{\omega_z}{\tilde{\omega}_c}(a_{c,i} + a_{c,i}^\dagger + a_{m,i} + a_{m,i}^\dagger)(a_{c,j} + a_{c,j}^\dagger + a_{m,j} + a_{m,j}^\dagger) \\ &- \hbar\xi_{i,j}\frac{\omega_z}{\tilde{\omega}_c}(a_{c,i} - a_{c,i}^\dagger - a_{m,i} + a_{m,i}^\dagger)(a_{c,j} - a_{c,j}^\dagger - a_{m,j} + a_{m,j}^\dagger), \end{aligned} \tag{17}$$

where the coupling strength

$$\xi_{i,j} \equiv \frac{e^2}{8\pi\epsilon_0 m_e \omega_z d_{i,j}^3} = \frac{1}{\hbar}\frac{e^2}{4\pi\epsilon_0 d_{i,j}}\left(\frac{\Delta z}{d_{i,j}}\right)^2 \tag{18}$$

amounts to the Coulomb energy times the square of the ratio between the axial ground state amplitude and the inter-particle distance. We have observed that each oscillator,

axial, cyclotron and magnetron, is characterized by a typical resonance frequency. As a consequence, the coupling introduced by the Coulomb interaction between the degrees of freedom of different electrons is effective only for almost resonant oscillators. Therefore, in Eq. (17) the terms that couple the cyclotron and magnetron motion of the two particles give negligible effects. Furthermore, we are not interested in the coupling between the magnetron motion of different electrons, since this mode is essentially decoupled from the other degrees of freedom. Hence, disregarding the magnetron motion, the part of the system Hamiltonian describing the Coulomb repulsion between electrons i and j reduces to

$$H_{i,j}^{C} \simeq -2\hbar\xi_{i,j}(a_{z,i}+a_{z,i}^{\dagger})(a_{z,j}+a_{z,j}^{\dagger})+2\hbar\xi_{i,j}\frac{\omega_z}{\tilde{\omega}_c}\left(a_{c,i}a_{c,j}^{\dagger}+a_{c,i}^{\dagger}a_{c,j}\right). \tag{19}$$

In the case of a linear array of electrons, trapped in a direction orthogonal to the magnetic field, i.e. along the x axis as shown in Fig. 1(b), we can derive a similar expression for the Coulomb coupling

$$\begin{aligned} H_{i,j}^{C} &\simeq \hbar\xi_{i,j}(a_{z,i}+a_{z,i}^{\dagger})(a_{z,j}+a_{z,j}^{\dagger})-\hbar\xi_{i,j}\frac{\omega_z}{\tilde{\omega}_c}\Big[a_{c,i}a_{c,j}^{\dagger}+a_{c,i}^{\dagger}a_{c,j} \\ &+ 3(a_{c,i}a_{c,j}+a_{c,i}^{\dagger}a_{c,j}^{\dagger})\Big]. \end{aligned} \tag{20}$$

We emphasize that in the case of Eq. (19), referring to the vertical array of traps, the coupling between the cyclotron oscillators of different electrons represent a swapping of excitations, which basically conserves energy. The only terms that survive involve the creation of a quantum of excitation at the site j at the expense of the destruction of a quantum of excitation at the site i and viceversa. In the case of an horizontal arrangement of traps, Eq. (20), this is, in general, no longer true. Indeed, even though the leading terms are preserving the energy of the two coupled cyclotron oscillators, we also note the presence of rapidly rotating terms which involve the simultaneous creation and annihilation of two excitations. However if $\xi_{i,j}(\omega_z/\tilde{\omega}_c) \ll \omega_c$ the effects of these rapidly rotating terms are negligible (RWA) and the Hamiltonian (20) becomes

$$H_{i,j}^{C} \simeq \hbar\xi_{i,j}(a_{z,i}+a_{z,i}^{\dagger})(a_{z,j}+a_{z,j}^{\dagger})-\hbar\xi_{i,j}\frac{\omega_z}{\tilde{\omega}_c}\left(a_{c,i}a_{c,j}^{\dagger}+a_{c,i}^{\dagger}a_{c,j}\right). \tag{21}$$

We also note that Eqs. (19) and (21) exhibit alternating signs in front of the coupling terms. As we will see in the next section, this results in a different kind, ferromagnetic or antiferromagnetic, of the effective spin-spin interaction.

III. EFFECTIVE SPIN-SPIN COUPLING

In the previous section we have seen that the magnetic gradient induces, for each particle of the array, a coupling between the spatial and the spin motions. This interaction, mediated by the Coulomb repulsion between the electrons, gives rise to an effective spin-spin coupling between different particles [11]. This effect can be brought to light by making an appropriate unitary transformation on the Hamiltonian of the system [19]. We seek a transformation that formally removes, in the single particle Hamiltonian, the coupling between the internal and the external degrees of freedom of each electron. Hence, we transform the Hamiltonian, Eq. (1), as $H' = e^S H e^{-S}$ with

$$S \;=\; \frac{g}{4}\,\varepsilon\sum_{i=1}^{N}\left[\sigma_i^z(a_{z,i}^\dagger - a_{z,i}) + \frac{\omega_z}{\omega_a}\sqrt{\frac{\omega_z}{\tilde{\omega}_c}}\left(\sigma_i^{(-)}a_{c,i}^\dagger - \sigma_i^{(+)}a_{c,i}\right)\right], \tag{22}$$

where $\omega_a \equiv \omega_s - \omega_c$ is the anomaly frequency. The unitary transformation changes the operators, to the first order in ε, in the following way

$$a_{z,i} \;\longrightarrow\; a_{z,i} - \frac{g}{4}\varepsilon\,\sigma_i^z, \tag{23}$$

$$a_{c,i} \;\longrightarrow\; a_{c,i} - \frac{g}{4}\varepsilon\frac{\omega_z}{\omega_a}\sqrt{\frac{\omega_z}{\tilde{\omega}_c}}\,\sigma_i^{(-)}, \tag{24}$$

$$\sigma_i^z \;\longrightarrow\; \sigma_i^z + \frac{g}{2}\varepsilon\frac{\omega_z}{\omega_a}\sqrt{\frac{\omega_z}{\tilde{\omega}_c}}\left(\sigma_i^{(+)}a_{c,i} + \sigma_i^{(-)}a_{c,i}^\dagger\right), \tag{25}$$

$$\sigma_i^{(+)} \;\longrightarrow\; \sigma_i^{(+)} + \frac{g}{2}\varepsilon\,\sigma_i^{(+)}(a_{z,i}^\dagger - a_{z,i}) - \frac{g}{4}\varepsilon\frac{\omega_z}{\omega_a}\sqrt{\frac{\omega_z}{\tilde{\omega}_c}}\,\sigma_i^z a_{c,i}^\dagger\,. \tag{26}$$

To derive the expressions above we made use of the expansion

$$e^{\eta A}Be^{-\eta A} = B + \eta[A,B] + \frac{\eta^2}{2!}[A,[A,B]] + \frac{\eta^3}{3!}[A,[A,[A,B]]] + \ldots, \tag{27}$$

where A and B are two noncommuting operators and η is a parameter.

The single electron part, Eq. (15), of the system Hamiltonian can be written, after applying the unitary transformation, as

$$H_i^{\prime NC} \simeq -\hbar\omega_m a_{m,i}^\dagger a_{m,i} + \hbar\omega_c a_{c,i}^\dagger a_{c,i} + \hbar\omega_z a_{z,i}^\dagger a_{z,i} + \frac{\hbar}{2}\omega_s\sigma_i^z, \tag{28}$$

where we have neglected second and higher order terms in ε, which in the cases relevant to the present analysis is of the order of 10^{-2}. Nevertheless, these extra terms are derived in Appendix A and their influence on the performances of the system is discussed in Sec. IV.

Let us now turn to the Coulomb part of the system Hamiltonian. The first term in Eq. (19) becomes

$$-2\hbar\xi_{i,j}\left(a_{z,i}+a_{z,i}^{\dagger}-\frac{g}{2}\varepsilon\sigma_i^z\right)\left(a_{z,j}+a_{z,j}^{\dagger}-\frac{g}{2}\varepsilon\sigma_j^z\right). \tag{29}$$

Expression (29) contains a term which represents an effective spin-spin coupling between different electrons in the array. This effect was already pointed out in Ref. [11]. Moreover, we note that the unitary transformation enforces a coupling between the axial motion of the j-th electron and the spin of the i-th electron. This effect is smaller of a factor $\xi_{i,j}/\omega_z \ll 1$ than the corresponding coupling [see Eq. (15)] between internal (spin) and external (axial motion) degrees of freedom of the same particle. The error introduced by neglecting these terms is estimated in the Appendix.

The remaining term in Hamiltonian (19) transforms into

$$\begin{aligned} & 2\hbar\xi_{i,j}\frac{\omega_z}{\tilde{\omega}_c}\left(a_{c,i}-\frac{g}{4}\varepsilon\frac{\omega_z}{\omega_a}\sqrt{\frac{\omega_z}{\tilde{\omega}_c}}\sigma_i^{(-)}\right)\left(a_{c,j}^{\dagger}-\frac{g}{4}\varepsilon\frac{\omega_z}{\omega_a}\sqrt{\frac{\omega_z}{\tilde{\omega}_c}}\sigma_j^{(+)}\right) \\ + \ & 2\hbar\xi_{i,j}\frac{\omega_z}{\tilde{\omega}_c}\left(a_{c,i}^{\dagger}-\frac{g}{4}\varepsilon\frac{\omega_z}{\omega_a}\sqrt{\frac{\omega_z}{\tilde{\omega}_c}}\sigma_i^{(+)}\right)\left(a_{c,j}-\frac{g}{4}\varepsilon\frac{\omega_z}{\omega_a}\sqrt{\frac{\omega_z}{\tilde{\omega}_c}}\sigma_j^{(-)}\right). \end{aligned} \tag{30}$$

From Eq. (30) we see that the unitary transformation produces the term

$$\hbar\xi_{i,j}\varepsilon^2\frac{g^2}{8}\frac{\omega_z^4}{\omega_a^2\tilde{\omega}_c^2}\left(\sigma_i^{(-)}\sigma_j^{(+)}+\sigma_i^{(+)}\sigma_j^{(-)}\right)=\hbar\xi_{i,j}\varepsilon^2\frac{g^2}{16}\frac{\omega_z^4}{\omega_a^2\tilde{\omega}_c^2}(\sigma_i^x\sigma_j^x+\sigma_i^y\sigma_j^y), \tag{31}$$

which represents a direct coupling between the spin motion of different particles. Also in this case, there are additional terms in expression (30), that couple the spin of an electron to the cyclotron motion of the other electrons in the chain. In comparison with the spin-cyclotron interaction for the same particle [see Eq. (15)], this coupling is reduced of a factor $\xi_{i,j}\omega_z/\tilde{\omega}_c\omega_a$, which, in the range of the parameters considered here, is typically much less than one. For an estimate of the errors introduced by these terms we refer to the Appendix.

Hence, summarizing the results of our derivation, we have an effective spin-spin coupling between the electrons with the spatial dynamics substantially decoupled from the spin dynamics. Consequently the spin part of the system Hamiltonian can be written, in the case of a linear array of electrons along the z axis, as

$$H_s' \simeq \sum_{i=1}^{N}\frac{\hbar}{2}\omega_s\sigma_i^z-\hbar\sum_{i>j}^{N}(2J_{i,j}^z\sigma_i^z\sigma_j^z-J_{i,j}^{xy}\sigma_i^x\sigma_j^x-J_{i,j}^{xy}\sigma_i^y\sigma_j^y), \tag{32}$$

where

$$J_{i,j}^{z} = \left(\frac{g}{2}\right)^2 \xi_{i,j}\varepsilon^2 = \left(\frac{g}{2}\right)^2 \frac{\hbar e^4 b^2}{16\pi\epsilon_0 m_e^4 \omega_z^4 d_{i,j}^3}, \tag{33}$$

$$J_{i,j}^{xy} = \left(\frac{g}{4}\right)^2 \xi_{i,j}\varepsilon^2 \frac{\omega_z^4}{\omega_a^2 \tilde{\omega}_c^2} \simeq 10^6 \left(\frac{g}{4}\right)^2 \frac{\hbar e^4 b^2}{16\pi\epsilon_0 m_e^4 \omega_c^4 d_{i,j}^3}. \tag{34}$$

In Eq. (34) we used the relations $\omega_a \simeq 10^{-3}\omega_c$ and $\tilde{\omega}_c \simeq \omega_c$. We obtain a spin-spin interaction that is antiferromagnetic (ferromagnetic) if it is transmitted by the cyclotron (axial) motion.

The situation is completely different when the linear array of electrons is aligned along the x axis

$$H_s' \simeq \sum_{i=1}^{N} \frac{\hbar}{2}\omega_s \sigma_i^z + \frac{\hbar}{2}\sum_{i<j}^{N}\left(2J_{i,j}^{z}\sigma_i^z\sigma_j^z - J_{i,j}^{xy}\sigma_i^x\sigma_j^x - J_{i,j}^{xy}\sigma_i^y\sigma_j^y\right). \tag{35}$$

In this case, the sign of the Heisenberg like coupling is reversed. The ferromagnetic (antiferromagnetic) interaction is associated to the cyclotron (axial) motion. Similar results were also found in the case of ions, in a Paul trap, driven by six counterpropagating laser beams [14].

IV. A CHANNEL FOR QUANTUM COMMUNICATION

The Hamiltonians (32) and (35) describe a system of N spins coupled through Heisenberg-like interactions. These Hamiltonians can transmit an unknown spin state, from the electron placed at one end of the linear array, to the electron placed at the other end of the array. The remarkable fact is that this quantum information transfer is realized only by means of the free dynamical evolution of the system, without requiring any external action by the experimenter during the transfer.

Therefore, let us analyze the potentialities of our system as a quantum communication channel. In our scheme, the dependence of the spin-spin coupling strength on the system parameters is shown in Eqs. (33) and (34). In particular, $J_{i,j}^{z}$, $J_{i,j}^{xy}$ are proportional to $1/d_{i,j}^3$, that is they decrease with the distance between the particles i and j according to the dipolar decay law. They also depend on the applied magnetic field gradient and on the characteristic frequencies of the electron motion. More specifically, the value of $J_{i,j}^{xy}$ ($J_{i,j}^{z}$) depends on the cyclotron (axial) frequency ω_c (ω_z). As a consequence of this fact we can neglect $J_{i,j}^{xy}$ with respect to $J_{i,j}^{z}$ when the value of the ratio ω_c/ω_z is sufficiently large, as in the case considered

in [11]. Differently, in this paper, we choose smaller values for the ratio ω_c/ω_z (generally about 20 or less), so that $J^{xy}_{i,j}$ is of the same order of magnitude of $J^{z}_{i,j}$ or even larger. Indeed, one can easily check, from Eqs. (33) and (34), that when $\omega_c/\omega_z \simeq 18.8$ it is possible to obtain an isotropic Heisenberg-like interaction with $2J^{z}_{i,j} = J^{xy}_{i,j}$.

Generally the time required to transfer a qubit, encoded in the spin state, along a Heisenberg chain depends on the values of $J^{xy}_{i,j}$, so that the larger the value of $J^{xy}_{i,j}$ the faster the transfer. Indeed, the state transfer time t_{ex} in a Heisenberg chain, consisting of just two spins, is equal to

$$t_{ex} \equiv \frac{\pi}{4J^{xy}}. \tag{36}$$

We assume that the particles in our linear array are equally spaced with $d \equiv d_{i,i+1}$ and $J^{xy} \equiv J^{xy}_{i,i+1}$. From Eq. (34) we see that $J^{xy} \propto b^2/(\omega_c^4 d^3)$. Hence, to speed up the transfer process we have to miniaturize the system, to increase the strength of the magnetic field gradient and to reduce the cyclotron frequency. However, the value of the cyclotron frequency ω_c, depending on the confining magnetic field, cannot be decreased at will, since it should be sufficiently large to cool the cyclotron motion to its ground state. For example, at the trap temperature of 80 mK [17] it is sufficient a cyclotron frequency of the order of 10 GHz. The inter-particle distance d depends on the level of miniaturization of the trap. We consider d varying from few microns to 50 μm. Finally stronger local magnetic gradients are, in general, achievable by reducing the micro-trap size.

Essentially, the effective Heisenberg-like Hamiltonians, Eqs. (32) and (35), have been obtained by taking two steps: we applied an appropriate unitary transformation and then disregarded the residual coupling between the different degrees of freedom. Both these steps, in general, introduce errors which reduce the accuracy of our system in reproducing an array of particles interacting according to the Heisenberg model. In particular, we neglected terms representing a residual coupling between the spin and the motional degrees of freedom, as well as between the different spatial oscillators. In Appendix A, we analyse in detail the role of each of these terms. Here, we only present the most relevant part of this interaction

$$H_r \simeq \varepsilon^2 \sum_{i=1}^{N} \hbar\omega_z \left[\frac{\omega_z^2}{4\omega_c\omega_a} - \frac{\omega_z^2}{4\omega_c^2} a^\dagger_{m,i} a_{m,i} + \left(\frac{\omega_z^2}{4\omega_c\omega_a} - 1 \right) a^\dagger_{c,i} a_{c,i} \right] \sigma_i^z, \tag{37}$$

which affects the spin frequency, introducing a dependence on the cyclotron and magnetron motion. As a consequence each particle acquires a different spin frequency with a finite linewidth due to the thermal state of the motional degrees of freedom.

In order to know how precisely our model can simulate an ideal Heisenberg system we introduce the system fidelity

$$\mathcal{F} \equiv \langle \psi_f | \mathrm{Tr}_{ext}[\rho(t)] | \psi_f \rangle, \tag{38}$$

where

$$|\psi_f\rangle \equiv e^{-\frac{i}{\hbar} H_s t} |\psi_0\rangle, \tag{39}$$

with $|\psi_0\rangle$ being the initial state of the spin chain and H_s is the Heisenberg Hamiltonian Eq. (32). The operator $\rho(t)$ in Eq. (38) represents the density operator of the electron chain, including the motional degrees of freedom, evolved at the time t according to the full Hamiltonian of the system Eq. (1). We also assume that initially the axial, cyclotron, and magnetron motions are prepared in thermal mixtures with, respectively, an average excitation number $\bar{k}$, $\bar{n}$ and $\bar{l}$. The reduced density operator, describing the spin state, is then obtained by tracing over the spatial modes of the electrons.

The system fidelity can be analytically calculated. The details are provided in Appendix A. In general, the fidelity can be written as

$$\mathcal{F} = 1 - \mathcal{E}_r - \varepsilon^2 \mathcal{E}_S, \tag{40}$$

where $\mathcal{E}_r$ and $\mathcal{E}_S$ represent, respectively, the errors due to the residual coupling, Eq. (37), and to the canonical transformation. In the simplest case of just two electrons, we find

$$\mathcal{E}_r = 1 - \sum_{n_1, l_1} \sum_{n_2, l_2} P_{\bar{n}}(n_1) P_{\bar{l}}(l_1) P_{\bar{n}}(n_2) P_{\bar{l}}(l_2) \left[\mathcal{F}_d \left(\frac{\delta_s(n_1, l_1, n_2, l_2)}{4 J^{xy}} \right) \right], \tag{41}$$

with $P_{\overline{m}}(m)$, Eq. (A7), being the occupation probability for the mth Fock state,

$$\mathcal{F}_d(\zeta) = \frac{1}{3} \left[1 + \frac{\cos(\frac{\zeta\pi}{2}) \sin(\frac{\pi}{2}\sqrt{1+\zeta^2})}{\sqrt{1+\zeta^2}} + \frac{\sin^2(\frac{\pi}{2}\sqrt{1+\zeta^2})}{1+\zeta^2} \right] \tag{42}$$

and

$$\delta_s \equiv \varepsilon^2 \omega_z \left[\left(\frac{\omega_z^2}{2\omega_c \omega_a} - 2 \right) (n_2 - n_1) - \frac{\omega_z^2}{2\omega_c^2} (l_2 - l_1) \right] \tag{43}$$

being the detuning between the two spin frequencies. The fidelity decreases because of this finite detuning, which is determined by the thermal state of the cyclotron and magnetron oscillators. Indeed, the error, Eq. (41), vanishes in the ideal case of zero detuning $\delta_s = 0$. Also the error due to the canonical transformation

$$\mathcal{E}_S = \frac{1}{3}(2\bar{k} + 1) + \frac{\omega_z^3}{6\omega_a^2 \omega_c}(5\bar{n} + 1) + \frac{\omega_z^3}{6(\omega_s - \omega_m)^2 \omega_c}(5\bar{l} + 4) \tag{44}$$

	A				B	
d (μm)	50	30	10	3	10	3
$\omega_z/2\pi$(MHz)	490	490	490	1200	730	4500
b(T/m)	350	600	1800	1800	1100	1100
$\bar{l}$	0.01	0.1	2	50	0.15	1
J^{xy} (kHz)	0.01	0.14	35	1300	2.5	100

TABLE I: Table showing the values of the axial frequency $\omega_z/2\pi$, the magnetic gradient b, the average magnetron excitation number $\bar{l}$ and the coupling strength J^{xy} for different choices of the nearest neighbor distance d. In case A (B) we have $\mathcal{F} = 0.99$ ($\mathcal{F} = 0.999$) and $\omega_c/2\pi = 8$ GHz ($\omega_c/2\pi = 11$ GHz). We suppose that the axial and cyclotron motion are thermalized with the trap environment at the temperature of 80 mK.

becomes larger when the electron motion is relatively *hot*. From Eq. (44), we see that this error is proportional to the average excitation numbers $\bar{k}$, $\bar{n}$, and $\bar{l}$. Therefore, to increase the system fidelity it is essential to cool, possibly to the ground state, the electron motion. This comes automatically for the cyclotron oscillator, when the trap is at a temperature below 1 K, whereas the cooling of the axial and magnetron oscillators requires appropriate techniques [16, 18].

We present a number of cases in Table I, when the fidelity approaches the value one. We see that, for $\mathcal{F} = 0.99$ ($\mathcal{F} = 0.999$) and the inter-particle distance d ranging from 50 μm (10 μm) to few microns, we have a coupling constant J^{xy} in the range 10 Hz $\div$ 1.3 MHz (2.5 kHz $\div$ 100 kHz). For example, in the case of $d = 10$ μm we obtain $J^{xy} = 35$ kHz by taking a cyclotron frequency $\omega_c/2\pi = 8$ GHz, an axial frequency $\omega_z/2\pi = 490$ MHz, and a magnetic gradient $b = 1800$ T/m.

We also recall that the decoherence time of the spin state as well as the heating time of the spatial motions, estimated according to the model described in [10, 21, 22], is much longer than the transfer time t_{ex}. This remains true also for moderate values of the coupling strength J^{xy}, thus allowing the transmission of the qubit state across the chain within the decoherence time of the system.

Finally we note that our system offers the possibility, in principle, to apply arbitrary site specific changes to its parameters, such as the inter-particle distance, the magnetic gradient

strength, and the magnetic field magnitude. Hence, as suggested in [7, 8], by means of these local modifications one can optimize the transmission rate and the fidelity of the chain.

V. CONCLUSIONS

In this paper we presented a scheme for implementing a spin chain with long range interactions by means of a linear array of electrons confined in micro-Penning traps. Both antiferromagnetic and ferromagnetic Heisenberg-like systems can be realized using a local magnetic field gradient, mediated by the electrostatic interaction between the trapped particles. In particular, we derived an analytical formula for the strength of the spin-spin coupling, which determines the transmission rate of the channel, as a function of the relevant system parameters like the inter-particle distance, the cyclotron frequency, and the value of the applied magnetic gradient. In our analysis we also estimated the fidelity of the system in reproducing an effective Heisenberg chain, by taking into account the effects produced by the coupling between the different degrees of freedom of the particles. We found that the fidelity depends on the frequency and the amplitude of the spatial motion of the particles. In general, higher values of the fidelity are obtained for smaller values of the spatial motion amplitudes and for larger values of the detuning between the characteristic trapping frequencies. The numerical estimates, calculated for an inter-particle distance d varying from 50 μm to few microns, give a spin-spin coupling strength J^{xy} in the range 10 Hz $\div$ 1.3 MHz with a fidelity of 99%. Even in the case of a relatively weak coupling constant, the transmission of the qubit state from one end to the other of the chain takes place well within the decoherence time of the system. Moreover, the geometry of the system offers the possibility to apply arbitrary site-specific changes of its parameters in order to optimize the transmission rate and the fidelity of the quantum channel.

In conclusion, an array of electrons confined in micro-Penning traps lends itself to implement, within the reach of current technology, quantum channels with high accuracy and sufficiently large transmission rates. Furthermore, the versatility of our scheme allows one to simulate also more general spin systems, in one and two dimensions, thus paving the way towards the observation of quantum phase transitions.

APPENDIX A: FIDELITY

In this appendix we provide a brief description of the approach adopted to estimate the fidelity, as defined in Eq. (38). Our starting point is the complete single electron Hamiltonian Eq. (14). In order to remove from this Hamiltonian, to the first order in ε, the coupling between the different particle motions, we should apply a unitary transformation which takes into account also the magnetron oscillator

$$\begin{aligned} S &= \varepsilon \sum_{i=1}^{N} \left[\left(\frac{g}{4}\sigma_i^z + a_{c,i}^\dagger a_{c,i} + \frac{\omega_m}{\omega_c} a_{m,i}^\dagger a_{m,i} \right) (a_{z,i}^\dagger - a_{z,i}) + \frac{g}{4}\frac{\omega_z}{\omega_a}\sqrt{\frac{\omega_z}{\tilde{\omega}_c}} \left(\sigma_i^{(-)} a_{c,i}^\dagger - \sigma_i^{(+)} a_{c,i} \right) \right. \\ &+ \left. \frac{g}{4}\frac{\omega_z}{\omega_s - \omega_m}\sqrt{\frac{\omega_z}{\tilde{\omega}_c}} \left(\sigma_i^{(-)} a_{m,i} - \sigma_i^{(+)} a_{m,i}^\dagger \right) \right]. \end{aligned} \tag{A1}$$

This unitary transformation represents a generalization of the transformation Eq. (22), since it encompasses all the degrees of freedom of the particles.
From the definition of the fidelity, Eq. (38), we can write [14]

$$\mathcal{F} = \langle \psi_f | \mathrm{Tr}_{ext}[e^{-S} e^{-\frac{i}{\hbar}H' t} e^{S} \rho(0) e^{-S} e^{\frac{i}{\hbar}H' t} e^{S}] | \psi_f \rangle, \tag{A2}$$

where

$$H' \equiv H_{ext} + H_s + H_r, \tag{A3}$$

with

$$H_{ext} = \sum_{i=1}^{N} \left(-\hbar\omega_m a_{m,i}^\dagger a_{m,i} + \hbar\omega_c a_{c,i}^\dagger a_{c,i} + \hbar\omega_z a_{z,i}^\dagger a_{z,i} \right) \tag{A4}$$

being the Hamiltonian describing the uncoupled external dynamics of the particles. The spin Hamiltonian H_s is given in Eq. (32), whereas H_r includes the residual coupling between the spin and the spatial degrees of freedom

$$\begin{aligned} H_r &\simeq \varepsilon^2 \sum_{i=1}^{N} \hbar\omega_z \left\{ \left[\frac{\omega_z^2}{4\omega_c\omega_a} - \frac{\omega_z^2}{4\omega_c^2} a_{m,i}^\dagger a_{m,i} + \left(\frac{\omega_z^2}{4\omega_c\omega_a} - 1 \right) a_{c,i}^\dagger a_{c,i} \right] \sigma_i^z \right. \\ &- \frac{1}{2}\sqrt{\frac{\omega_z}{\omega_c}} \left(a_{z,i}^\dagger - a_{z,i} \right) \left[\sigma_i^{(+)} a_{c,i} - \sigma_i^{(-)} a_{c,i}^\dagger + \frac{3}{2} \left(\sigma_i^{(+)} a_{m,i}^\dagger - \sigma_i^{(-)} a_{m,i} \right) \right] \\ &+ \left. \frac{\omega_z^2}{8\omega_c\omega_a} \left(a_{c,i} a_{m,i} + a_{c,i}^\dagger a_{m,i}^\dagger \right) \sigma_i^z \right\} + \varepsilon \sum_{i \neq j}^{N} \hbar\xi_{i,j} \left[g \left(a_{z,i} + a_{z,i}^\dagger \right) \sigma_j^z \right. \\ &- \left. \frac{g}{2} \left(\frac{\omega_z}{\tilde{\omega}_c} \right)^{\frac{3}{2}} \frac{\omega_z}{\omega_a} \left(a_{c,i} \sigma_j^{(+)} + a_{c,i}^\dagger \sigma_j^{(-)} \right) \right]. \end{aligned} \tag{A5}$$

We assume that initially the cyclotron, axial, and magnetron oscillators are in a thermal mixture, each one represented by the usual density operator

$$\rho_{th} = \sum_{m=0}^{+\infty} P_{\overline{m}}(m)|m\rangle\langle m|, \tag{A6}$$

with [23]

$$P_{\overline{m}}(m) \equiv \left(\frac{1}{1+\overline{m}}\right)\left(\frac{\overline{m}}{1+\overline{m}}\right)^m \tag{A7}$$

being the occupation probability of the mth Fock state of a harmonic oscillator with average excitation number $\overline{m}$. The initial spin state of the chain is

$$|\psi_0\rangle \equiv \left(\cos\frac{\theta}{2}|\downarrow\rangle_1 + e^{i\phi}\sin\frac{\theta}{2}|\uparrow\rangle_1\right)|\downarrow\rangle_2\ldots|\downarrow\rangle_N. \tag{A8}$$

The information is stored in the state of the first qubit and should be transmitted to the opposite end of the chain, to the Nth spin. Therefore, the density operator of the system at time $t = 0$ is

$$\rho(0) \equiv \rho^{(spin)} \otimes \rho^{(ext)}. \tag{A9}$$

The ideal final state of the spin chain is represented by the state vector

$$|\psi_f\rangle = \exp\left(-\frac{i}{\hbar}H_s t\right)|\psi_0\rangle, \tag{A10}$$

which is obtained from the initial spin state, Eq. (A8), when the system is described by the Heisenberg Hamiltonian H_s, Eq. (32).

Now to calculate the value of the fidelity, we make an expansion of Eq. (A2) in powers of S and consider terms up to the second order in ε

$$\begin{aligned}\mathcal{F} \simeq\ & \langle A\rho A^{-1}\rangle + \frac{1}{2}\langle A\rho A^{-1}S^2 + A\rho S^2 A^{-1} + AS^2\rho A^{-1} + S^2 A\rho A^{-1}\rangle \\ -\ & \langle A\rho SA^{-1}S - AS\rho A^{-1}S + AS\rho SA^{-1} + SA\rho A^{-1}S - SA\rho SA^{-1} + SAS\rho A^{-1}\rangle,\end{aligned} \tag{A11}$$

where we defined $A \equiv \exp[-(i/\hbar)H't]$, $\rho \equiv \rho(0)$ and $\langle\ldots\rangle \equiv \langle\psi_f|\mathrm{Tr}_{ext}[\ldots]|\psi_f\rangle$. The first order terms in S have been omitted since their contribution, after tracing over the spatial degrees of freedom, is zero.

In the absence of the residual couplings, contained in the Hamiltonian H_r, the spin chain evolution is unaffected by the thermal state of the motional degrees of freedom, because $[H_{ext}, H_s] = 0$. This leads to $\langle A\rho A^{-1}\rangle = 1$. The corrections to the fidelity come both from the presence of H_r and from the canonical transformation, represented by the remaining ten

terms of Eq. (A11). In order to separate the two effects, we first evaluate the impact of the unitary transformation when $H' \simeq H_{ext} + H_s$. This greatly simplifies the procedure and allows to achieve an analytical expression for the fidelity

$$\mathcal{F} \simeq \langle A\rho A^{-1}\rangle - \varepsilon^2 \mathcal{E}_S, \tag{A12}$$

where

$$\begin{aligned}
\mathcal{E}_S \;\simeq\; & \sum_{i=1}^{N} \left\{ \left[\left(\frac{g}{4}\right)^2 \left(2 - |\langle\sigma_i^z\rangle_0|^2 - |\langle\sigma_i^z\rangle_f|^2\right) + \frac{g}{2}\left(\langle\sigma_i^z\rangle_0 - \langle\sigma_i^z\rangle_f\right)\left(\bar{n} + \frac{\omega_m}{\omega_c}\bar{l}\right)\right](2\bar{k}+1) \right. \\
+ & \left(\frac{g}{4}\right)^2 \frac{\omega_z}{\tilde{\omega}_c}\left[\frac{\omega_z^2}{\omega_a^2}\left(2\bar{n}+1+\langle\sigma_i^z\rangle_0\right) + \frac{\omega_z^2}{(\omega_s-\omega_m)^2}\left(2\bar{l}+1-\langle\sigma_i^z\rangle_0\right)\right. \\
- & \left.\left.\left(\frac{\omega_z^2}{\omega_a^2}(2\bar{n}+1) + \frac{\omega_z^2}{(\omega_s-\omega_m)^2}(2\bar{l}+1)\right)\left(\langle\sigma_i^{(-)}\rangle_0\langle\sigma_i^{(+)}\rangle_0 + \langle\sigma_i^{(-)}\rangle_f\langle\sigma_i^{(+)}\rangle_f\right)\right]\right\}.
\end{aligned} \tag{A13}$$

The expectation values $\langle\ldots\rangle_0$ and $\langle\ldots\rangle_f$ are calculated, respectively, over the initial and final state of the spin chain. At the swapping time, when the state of the first spin has moved to the other end of the chain,

$$\sum_{i=1}^{N}\langle\sigma_i^z\rangle_0 \;=\; \sum_{i=1}^{N}\langle\sigma_i^z\rangle_f = -(N-1) - \cos\theta, \tag{A14}$$

$$\sum_{i=1}^{N}|\langle\sigma_i^z\rangle_0|^2 \;=\; \sum_{i=1}^{N}|\langle\sigma_i^z\rangle_f|^2 = N-1+\cos^2\theta, \tag{A15}$$

$$\sum_{i=1}^{N}\langle\sigma_i^{(-)}\rangle_0 \;=\; \sum_{i=1}^{N}\langle\sigma_i^{(-)}\rangle_f = \frac{e^{i\phi}}{2}\sin\theta, \tag{A16}$$

$$\sum_{i=1}^{N}\langle\sigma_i^{(+)}\rangle_0 \;=\; \sum_{i=1}^{N}\langle\sigma_i^{(+)}\rangle_f = \frac{e^{-i\phi}}{2}\sin\theta. \tag{A17}$$

(A18)

Moreover, after averaging over all the initial states in the Bloch sphere, i.e. evaluating $(1/4\pi)\int_0^\pi\int_0^{2\pi}\mathcal{E}_S\sin\theta\, d\theta\, d\phi$, we obtain

$$\mathcal{E}_S \simeq \frac{1}{3}(2\bar{k}+1) + \frac{1}{6}\frac{\omega_z}{\omega_c}\left[\frac{\omega_z^2}{\omega_a^2}\left(2\bar{n}+1+3(N-1)\bar{n}\right) + \frac{\omega_z^2}{(\omega_s-\omega_m)^2}\left(2\bar{l}+1+3(N-1)(\bar{l}+1)\right)\right], \tag{A19}$$

where $\bar{k}$, $\bar{n}$, and $\bar{l}$ denote, respectively, the average axial, cyclotron and magnetron excitation number. The expression (A19) gives the error due to the unitary transformation.

Let us consider now the effects of the Hamiltonian H_r, contained in the term $\langle A\rho A^{-1}\rangle$ of Eq. (A12). The residual couplings produce mainly two effects: they induce transitions

between the motional states of the electron and make the electron spin frequency depend on the state of the external degrees of freedom. Both these effects, as we will see, reduce the system fidelity.

The probability to observe transitions between the states of the electron motion can be easily estimated using a perturbative approach. Indeed, the probability for the transition $|\psi_m\rangle \rightarrow |\psi_n\rangle$ is not larger than roughly $4|\langle\psi_n|\Delta H|\psi_m\rangle|^2/(\hbar\omega_{nm})^2$, where ΔH is the perturbation, ω_{nm} the transition frequency and $|\psi_i\rangle$ the i-th eigenstate of the unperturbed Hamiltonian. In our case, the Hamiltonian H_r plays the role of ΔH and the terms, responsible for the transitions between electronic states, are in the last three lines of Eq. (A5). For example, the terms proportional to $a_{z,i}\sigma_i^{(+)}a_{c,i}$ induce transitions between the eigenstates $|n,k,l,\downarrow\rangle$ and $|n-1,k-1,l,\uparrow\rangle$ of the single electron Hamiltonian

$$H_0 = -\hbar\omega_m a_m^\dagger a_m + \hbar\omega_c a_c^\dagger a_c + \hbar\omega_z a_z^\dagger a_z + \frac{\hbar}{2}\omega_s\sigma^z \tag{A20}$$

with probability of the order of $\varepsilon^4[\omega_z^3/\omega_c(\omega_z-\omega_a)^2]kn$. A similar perturbative approach allows us to estimate also the transition probability due to the other terms of Eq. (A5). These probabilities, for the terms involving couplings between different motions of the same particle, are proportional to $\varepsilon^4/(\Delta\omega)^2$ where $\Delta\omega$ denotes the detuning between the electron oscillation frequencies. Hence, the error, in this case, is always negligible because is a correction of the fourth order in ε and, moreover, the characteristic frequencies of the electron motion are quite different from each other. Very small errors are also produced by the terms in Eq. (A5) involving the dynamics of different particles. In this case the transition probabilities are of the order of $\varepsilon^2(\xi_{i,j}/\omega_z)^2$ and $\varepsilon^2(\xi_{i,j}/\omega_z)^2(\omega_z/\omega_c)^3(\omega_z/\omega_a)^4$. Indeed, these values, for our choices of the system parameters, are negligible.

In addition to the state transitions, the residual couplings enforce a dependence of the spin frequency on the state of the particle motion. The correction $\Delta\omega_s$ to the spin frequency

$$\Delta\omega_s(n,l) \simeq \varepsilon^2\omega_z\left[\frac{\omega_z^2}{2\omega_c\omega_a} + \left(\frac{\omega_z^2}{2\omega_c\omega_a} - 2\right)n - \frac{\omega_z^2}{2\omega_c^2}l\right] \tag{A21}$$

depends on the cyclotron and magnetron excitations. Indeed, the constant shift proportional to $\omega_z^2/2\omega_c\omega_a$ equally affects all the spins in the chain and, therefore, does not introduce any detuning between the spin frequencies. This term can be taken into account by redefining the spin precession frequency ω_s. On the contrary, the last two terms of Eq. (A21) introduce a detuning between the spin frequencies along the chain, since the cyclotron and magnetron

oscillators are in a thermal mixture with fluctuating excitation numbers n and l. As a consequence each spin in the chain acquires a different frequency depending on the thermal state of the electron motion. This leads, as we will show, to a reduction of the system fidelity.

For the sake of simplicity, we restrict our analysis to the case of just two electrons in the chain. The corresponding Hamiltonian reads

$$H_{sd} = \frac{\hbar}{2}\omega_1\sigma_1^z + \frac{\hbar}{2}\omega_2\sigma_2^z + 2\hbar J^{xy}\left(\sigma_1^{(+)}\sigma_2^{(-)} + \sigma_1^{(-)}\sigma_2^{(+)}\right) - 2\hbar J^z\sigma_1^z\sigma_2^z, \tag{A22}$$

with $\omega_i = \omega_s + \Delta\omega(n_i, l_i)$. The unitary evolution of the system gives at the swapping time $t_{ex} = \pi/4J^{xy}$

$$|\downarrow\rangle_1|\downarrow\rangle_2 \;\rightarrow\; e^{2iJ^z t_{ex}} e^{\frac{i}{2}(\omega_1+\omega_2)t_{ex}}|\downarrow\rangle_1|\downarrow\rangle_2, \tag{A23}$$

$$\begin{aligned} |\uparrow\rangle_1|\downarrow\rangle_2 \;\rightarrow\;& e^{-2iJ^z t_{ex}}\left\{-\frac{i}{\sqrt{1+\zeta^2}}\sin\left(\frac{\pi}{2}\sqrt{1+\zeta^2}\right)|\downarrow\rangle_1|\uparrow\rangle_2\right. \\ &+\left.\left[\cos\left(\frac{\pi}{2}\sqrt{1+\zeta^2}\right) + \frac{i\zeta}{\sqrt{1+\zeta^2}}\sin\left(\frac{\pi}{2}\sqrt{1+\zeta^2}\right)\right]|\uparrow\rangle_1|\downarrow\rangle_2\right\}, \end{aligned} \tag{A24}$$

where $\zeta \equiv \delta_s/(4J^{xy})$ with

$$\delta_s(n_1, l_1, n_2, l_2) \equiv \omega_2 - \omega_1 = \varepsilon^2\omega_z\left[\left(\frac{\omega_z^2}{2\omega_c\omega_a} - 2\right)(n_2 - n_1) - \frac{\omega_z^2}{2\omega_c^2}(l_2 - l_1)\right]. \tag{A25}$$

Hence, by using the relations (A23) and (A24), we obtain the system fidelity

$$\mathcal{F}_d(\zeta) = \frac{1}{3}\left[1 + \frac{\cos(\frac{\zeta\pi}{2})\sin(\frac{\pi}{2}\sqrt{1+\zeta^2})}{\sqrt{1+\zeta^2}} + \frac{\sin^2(\frac{\pi}{2}\sqrt{1+\zeta^2})}{1+\zeta^2}\right] \tag{A26}$$

when the cyclotron and magnetron oscillators are in the states $|n_i, l_i\rangle$, with $i = 1, 2$. Consequently, in the case of a thermal mixture, the expression for the fidelity becomes

$$\mathcal{F} \equiv \sum_{n_1,l_1}\sum_{n_2,l_2} P_{\bar{n}}(n_1)P_{\bar{l}}(l_1)P_{\bar{n}}(n_2)P_{\bar{l}}(l_2)\left[\mathcal{F}_d\left(\frac{\delta_s(n_1, l_1, n_2, l_2)}{4J^{xy}}\right)\right]. \tag{A27}$$

We use this formula together with Eq. (A19) to numerically evaluate the fidelity of our system.

ACKNOWLEDGMENTS

This research was supported by the European Commission through the Specific Targeted Research Project *QUELE*, the Integrated Project FET/QIPC *SCALA*, and the Research

Training Network *CONQUEST*.

---

[1] S. Bose, Phys. Rev. Lett. **91**, 207901 (2003).

[2] T.J. Osborne, N. Linden, Phys. Rev. A **69**, 052315 (2004).

[3] M. Christandl, N. Datta, A. Ekert, and A. J. Landahl, Phys. Rev. Lett. **92**, 187902 (2004).

[4] M. Christandl, N. Datta, T. C. Dorlas, A. Ekert, A. Kay, and A. J. Landahl, Phys. Rev. A **71**, 032312 (2005).

[5] D.Burgarth and S. Bose, Phys. Rev. A **71**, 052315 (2005).

[6] D.Burgarth and S. Bose, New J. Phys. **7**, 135 (2005).

[7] A. Kay, Phys. Rev. A **73**, 032306 (2006).

[8] M. Avellino, A. J. Fisher and S. Bose, Phys. Rev. A **74**, 012321 (2006).

[9] G. Ciaramicoli, I. Marzoli, and P. Tombesi, Phys. Rev. Lett. **91**, 017901 (2003).

[10] G. Ciaramicoli, I. Marzoli, and P. Tombesi, Phys. Rev. A **70**, 032301 (2004).

[11] G. Ciaramicoli, F. Galve, I. Marzoli, and P. Tombesi, Phys. Rev. A **72**, 042323 (2005).

[12] J. R. Castrejón-Pita and R. C. Thompson Phys. Rev. A **72**, 013405 (2005).

[13] S. Stahl, F. Galve, J. Alonso, S. Djekic, W. Quint, T. Valenzuela, J. Verdù, M. Vogel, and G. Werth, Eur. Phys. J. D **32**, 139 (2005).

[14] D. Porras and J. I. Cirac, Phys. Rev. Lett. **92**, 207901 (2004).

[15] M. Drndić, C. S. Lee, and R. M. Westervelt, Phys. Rev. B **63**, 085321 (2001).

[16] L.S. Brown and G. Gabrielse, Rev. Mod. Phys. **58**, 233 (1986).

[17] S. Peil and G. Gabrielse, Phys. Rev. Lett. **83**, 1287 (1999).

[18] H.F. Powell, D.M. Segal, and R.C. Thompson, Phys. Rev. Lett. **89**, 093003 (2002).

[19] F. Mintert and Ch. Wunderlich, Phys. Rev. Lett. **87**, 257904 (2001); Ch. Wunderlich, *Laser Physics at the Limit* (Springer, Heidelberg, 2001), p. 261.

[20] J.D. Jackson, *Classical Electrodynamics*, 2nd ed., (Wiley, New York, 1975).

[21] Q.A. Turchette, D. Kielpinski, B.E. King, D. Leibfried, D.M. Meekhof, C.J. Myatt, M.A. Rowe, C.A. Sackett, C.S. Wood, W.M. Itano, C. Monroe, and D.J. Wineland, Phys. Rev. A **61**, 063418 (2000).

[22] C. Henkel, S. Pötting, and M. Wilkens, Appl. Phys. B: Lasers Opt. **69**, 379 (1999).

[23] W. P. Schleich, *Quantum Optics in Phase Space*, (Wiley-VCH, Berlin, 2001).